Torsion in the Cohomology of Arithmetic Subgroups

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May 4, 2020

Abstract

Let $\mathbf{G}$ be a semi-simple connected algebraic group over $\mathbb{Q}$ and Γ be an arithmetic subgroup of the group of real points $G = \mathbf{G}(\mathbb{R})$. Given a decreasing sequence of arithmetic subgroups $\{\Gamma_n\}$ of Γ, a recent conjecture has been made by Bergeron and Venkatesh on the limit as n goes to infinity of $\frac{\log |H_i(\Gamma_n, M)_{\text{tors}}|}{[\Gamma : \Gamma_n]}$, where M is a finite rank free $\mathbb{Z}$-module. It is conjectured that the limit depends on i and the deficiency of $\mathbf{G}$. In this expository paper, we introduce the reader to the literature concerning this problem, including Reidemeister torsion and analytic torsion, and recount recent developments made by Müller and others on specific cases of the conjecture.

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1 Introduction

Algebraic groups have long been of interest to mathematicians in a variety of different fields. Since they can be viewed in an algebraic and topological way, many tools lend themselves to study their shapes and aid in their classification. In this chapter, we introduce the foundations of algebraic group theory, which largely follows the material in [9].

We define an **algebraic group G** to be a variety endowed with the structure of a group, such that the multiplication map and the inversion map are morphisms of varieties (that is, the maps are given locally by polynomials). We define the **identity component** of $\mathbf{G}$ to be the irreducible component which contains the identity element, and we denote it as $\mathbf{G}^\circ$. We call an algebraic group **connected** if $\mathbf{G} = \mathbf{G}^\circ$.

In this paper, we will be focusing on algebraic groups which are affine varieties, which makes them **linear algebraic groups**. These are particularly nice to work with since

Theorem 1.1 ([9] Theorem 8.6): *If $\boldsymbol{G}$ is a linear algebraic group, then it is isomorphic to a closed subgroup (under the Zariski topology) of some $\mathrm{GL}_n(K)$, where K is an algebraically closed field.*

When $K = \mathbb{C}$, $\mathrm{GL}_n(K)$ is itself a Lie group, and being a closed subgroup, we can also view $\mathbf{G}$ as a Lie group too. With this perspective, we can view $\mathbf{G} < \mathrm{GL}_n(F)$ for some field F as a Lie group whenever the general linear group $\mathrm{GL}_n(F)$ is a Lie group.

Let $\mathbf{G}$ be an affine algebraic group over the field k; then if we let $I(\mathbf{G})$ denote the ideal of polynomials $f(t_1, \dots, t_r) \in k[t_1, \dots, t_r]$ that vanish on $\mathbf{G}$, we define the **affine algebra** of $\mathbf{G}$ to be $k[\mathbf{G}] := k[t_1, \dots, t_r]/I(\mathbf{G})$. We can define an action of $\mathbf{G}$ on $k[\mathbf{G}]$ like so; given an element $x \in \mathbf{G}$, the left translation $y \mapsto xy$ induces a map $(\lambda_x f)(y) = f(x^{-1}y)$ where $f \in k[\mathbf{G}]$. We call λ_x the **left translations of functions** by x, and note that it induces a group homomorphism

$$\lambda : \mathbf{G} \to \mathrm{GL}(k[\mathbf{G}])$$
$$\lambda : x \mapsto \lambda_x$$

One more key notion is that of **derivations** of $k[\mathbf{G}]$; these are k-linear maps $\delta : k[\mathbf{G}] \to k[\mathbf{G}]$ such that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in k[\mathbf{G}]$. It can be shown that these form a Lie algebra denoted $\mathrm{Der}(k[\mathbf{G}])$ where the Lie bracket is defined to be $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$. With these definitions in place we can define the **Lie algebra** of an algebraic group $\mathbf{G}$ to be the space of left invariant derivations

$$\mathrm{Lie}(\mathbf{G}) := \{\delta \in \mathrm{Der}(k[\mathbf{G}]) \,|\, \delta\lambda_x = \lambda_x\delta \qquad \forall x \in \mathbf{G}\}$$

Notice the similarities between this set up and the classical theory of Lie groups and Lie algebras.

In fact, in the same way that Lie groups give rise to Riemannian symmetric spaces, linear algebraic groups can too. Suppose that $\mathbf{G}$ is a semi-simple connected algebraic group over $\mathbb{Q}$, and K is a maximal compact subgroup of the group of real points $G := \mathbf{G}(\mathbb{R})$. Then the quotient $X := G/K$ can be shown to be a **Riemannian symmetric space**, that is, a connected Riemannian manifold X such that the isometry group $\mathrm{Isom}(X)$ acts transitively and such that there exists a $\phi \in \mathrm{Isom}(X)$ where $\phi^2 = \mathrm{id}$ and ϕ has an isolated fixed point [11]. Now when we take subgroups $\Gamma < G$, we are often interested in studying the resulting spaces $\Gamma\backslash X$. In particular, the most fruitful theory comes when we require $\Gamma\backslash X$ to be compact, or at least have finite volume. When $\Gamma < G$ is a discrete group (meaning there is a neighborhood around the identity element of G that doesn't intersect Γ), we call it **cocompact** if the space $\Gamma\backslash G$ is compact.

Example 1.1: Consider the affine algebraic group $G = \mathrm{SL}_2(\mathbb{R})$, which can be defined as a variety by $\{ad - bc = 1 \,|\, a, b, c, d \in \mathbb{R}\}$. From the Iwasawa decomposition, we have $\mathrm{SL}_2(\mathbb{R}) = KAN$ where

$$K = \mathrm{SO}(2), \qquad A = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right) \,\middle|\, a \in \mathbb{R}_{>0} \right\}, \qquad N = \left\{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \,\middle|\, n \in \mathbb{R} \right\}$$

Notice that up to homeomorphism, $K \cong S^1$, $A \cong \mathbb{R}_{>0}$ and $N \cong \mathbb{R}$. In particular, notice that $\mathrm{SL}_2(\mathbb{R})$ is connected and K is a compact subgroup. As in the general theory, we are interested in their quotient; from our decomposition above, it is clear that as groups there are isomorphisms $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{R}_{>0} \times \mathbb{R} \cong \mathfrak{H}$, where $\mathfrak{H} = \{x + iy \,|\, x, y \in \mathbb{R}\}$ is the complex upper half-plane. In fact, it can be shown that $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \cong \mathfrak{H}$ as a Riemannian symmetric space, and elements $g \in \mathrm{SL}_2(\mathbb{R})$ act on $z = x + iy \in \mathfrak{H}$ according to the linear fractional transformation

$$gz = \frac{az + b}{cz + d} = \frac{ac(x^2 + y^2) + x(ad + bc) + bd}{|cz + d|^2} + i\frac{y}{|cz + d|^2}$$

Next, we can show that $\mathfrak{H}$ comes endowed with a G-invariant measure: $\frac{dxdy}{y^2}$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, then we can view $gz = u + iv$ as a change of coordinates (where u and v are defined according to the equation above), and the Jacobian is equal to $\det(J) = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)$. Since the function $\frac{az+b}{cz+d}$ is holomorphic, it satisfies the Cauchy-Riemann equations, so we can simplify this expression to $\det(J) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$. Using our equations above, and the fact that $ad - bc = 1$, we can compute

$$\begin{aligned}
\det(J) &= \Big(\frac{(cx - cy + d)(cx + cy + d)}{(c^2y^2 + (cx + d)^2)^2}\Big)^2 + \Big(\frac{2cy(cx + d)}{(c^2x^2 + c^2y^2 + 2cdx + d^2)^2}\Big)^2 \\
&= \frac{1}{(c^2x^2 + c^2y^2 + 2cdx + d^2)^2} \\
&= \frac{1}{|cz + d|^4}
\end{aligned}$$

Let U be a subset of $\mathfrak{H}$ and $g \in \mathrm{SL}_2(\mathbb{R})$. Then when we recall that $v = \frac{y}{|cz+d|^2}$, our change of variables yields

$$\int\int_{gU} \frac{dudv}{v^2} = \int\int_U \frac{|cz + d|^4}{y^2} \det(J)dxdy = \int\int_U \frac{dxdy}{y^2}$$

Now consider the discrete subgroup $\mathrm{SL}_2(\mathbb{Z}) < G$. Although this group is not cocompact, we can show that $\mathrm{SL}_2(\mathbb{Z})\backslash G/K$ has finite volume. It is well known that the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ is

$$\mathcal{F} = \{z \in \mathfrak{H} \,|\, |z| \geq 1 \ \text{ and } \ -1/2 \leq \mathrm{Re}(z) \leq 1/2\}$$

(see Figure 1), and using the hyperbolic measure we have

$$\begin{aligned}
\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}) &= \int_{x+iy\in\mathcal{F}} d\mu \\
&= \int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^{\infty} \frac{dxdy}{y^2} \\
&= \int_{-1/2}^{1/2} \frac{dx}{\sqrt{1 - x^2}} \\
&= \arcsin(x)\Big|_{-1/2}^{1/2} \\
&= \frac{\pi}{3}
\end{aligned}$$

Finally, using $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, it can be shown from the action of the linear fractional transformation that the two vertical segments (from the Euclidean perspective) of $\mathcal{F}$ are identified

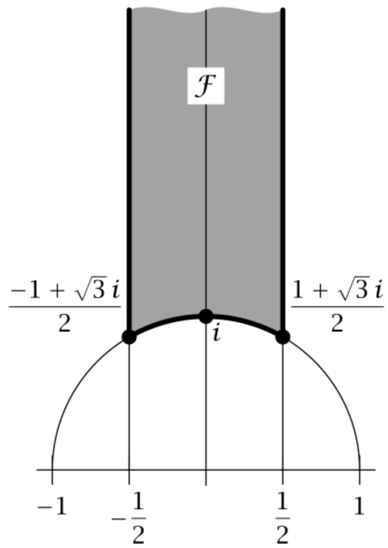

Figure 1: Fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$

together, and the two segments going from $e^{\pi i/3}$ and $e^{2\pi i/3}$ to i are identified together. After compactifying the space $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ by adding a point for the cusp at infinity, we get S^2. Likewise, we can consider other finite index subgroups $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ and analyze the resulting space $\Gamma\backslash G/K$, which after adding the appropriate cusps is a compact Riemann surface. Studying these modular curves is itself is an extremely fruitful area of research with ties into modular forms and elliptic curves (see for instance [5]).

2 The Bergeron-Venkatesh Conjecture

As with the notation of the previous chapter, let $\mathbf{G}$ be a semi-simple connected algebraic group over $\mathbb{Q}$, G be its group of real points, and $K < G$ be a maximal compact subgroup. Then we define the **deficiency** of $\mathbf{G}$ to be $\delta(\mathbf{G}) = \mathrm{rank}(G) - \mathrm{rank}(K)$, where the rank is the dimension of the maximal tori taken over $\mathbb{C}$. Because it makes sense to talk about the Lie algebra of an algebraic group, if $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of $G(\mathbb{R})$ and K respectively, then we also have $\delta(\mathbf{G}) = \mathrm{rank}(\mathfrak{g}_\mathbb{C}) - \mathrm{rank}(\mathfrak{k}_\mathbb{C})$. This is an equivalent definition because in the theory of linear algebraic groups, if G is semi-simple, then the maximal tori are exactly the Cartan subgroups [9].

As an example of computing the deficiency of algebraic groups, notice that $\mathrm{rank}(\mathrm{SL}_n(\mathbb{R})) = n-1$, which can be seen by examining the subgroup of diagonal matrices. A maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$ is $\mathrm{SO}(n)$, whose maximal torus is a block diagonal matrix where the blocks are 2-by-2 rotation matrices. Hence, we have $\mathrm{rank}(\mathrm{SO}(2n)) = \mathrm{rank}(\mathrm{SO}(2n+1)) = n$ and we deduce that $\delta(\mathrm{SL}_{2k}(\mathbb{R})) = k-1$ and $\delta(\mathrm{SL}_{2k+1}(\mathbb{R})) = k$ for all $k \in \mathbb{Z}_{>0}$.

With all of this notation in place, we are ready to state the main conjecture:

Conjecture 2.1 ([3] Conjecture 1.3): *Let $\Gamma > \Gamma_1 > \Gamma_2 > \dots$ be a decreasing sequence of cocompact congruence subgroups of G with $\cap_n \Gamma_n = \{1\}$, and let M be a a finite rank free $\mathbb{Z}$-module with a Γ-action. Then the limit*

$$\lim_{n\to\infty} \frac{\log |H_i(\Gamma_n, M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_n]}$$

exists for each i and is zero unless $\delta(\boldsymbol{G}) = 1$ and $i = \frac{\dim(G/K)-1}{2}$, in which case it is equal to a positive constant times the volume of $\Gamma \backslash G/K$.

Bergeron and Venkatesh's conjecture actually extends further depending on deficiency [3]:

- When $\delta(\mathbf{G}) = 0$, the limit should go to zero but $H(\Gamma_n, M \otimes \mathbb{Q})$ (the non-torsion) should still be large.
- When $\delta(\mathbf{G}) = 1$, there should be a lot of torsion in the homology groups, but $H(\Gamma_n, M \otimes \mathbb{Q})$ should be relatively small.
- When $\delta(\mathbf{G}) > 1$, neither the torsion nor the characteristic zero homology should grow exponentially.

There are numerous motivations for studying these cohomology groups. For one thing, there is a natural isomorphism between the cohomology of a torsion-free arithmetic subgroup $\Gamma < G$ and the cohomology of the associated Riemannian symmetric space G/K. This means that the phenomenon of torsion in cohomology likely has geometric meaning, which is of interest to differential geometers and topologists studying these Riemannian manifolds. Secondly, by a theorem of Franke, the cohomology of an algebraic group can be computed in terms of automorphic forms "of cohomological type," [6] hence there is a connection to the Langlands program. Finally, Avner Ash conjectured that Hecke eigenclasses in the group cohomology of these arithmetic subgroups with coefficients in $\mathbb{F}_p$ corresponds to Galois representations on $\mathbb{F}_p$ [1]. This is now a theorem of Scholze [18], and demonstrates that torsion in cohomology, even if it does not come from the reduction of cohomology classes with characteristic zero coefficients, has consequences in arithmetic geometry.

With these motivations in mind, Ash, Gunnells, McConnell, and Yasaki proposed an analogue conjecture to the case when Γ is not necessarily cocompact. Recall that the **cuspidal range** of an arithmetic group consists of the cohomoloical degrees where cuspidal automorphic forms contribute to the cohomology.

Conjecture 2.2 ([2] Conjecture 7.1): *Let Γ be any arithmetic group and $\{\Gamma_n\}$ be an infinite set of congruence subgroups of increasing prime level. Then*

$$\lim_{n\to\infty} \frac{\log |H^i(\Gamma_n, M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_n]}$$

should tend to the B-V limit when $\delta(\boldsymbol{G}) = 1$ and i is at the top of the cuspidal range.

Example 2.1: As a sanity check, let's check how the limit above behaves for the algebraic group $G = \mathrm{SL}_2(\mathbb{R})$. A maximal torus of G is $\left\{ \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix} \right) \mid r \in \mathbb{R} \right\}$, and hence $\mathrm{rank}(G) = 1$. As in Example 1.1, $K = \mathrm{SO}(2)$ is a maximal compact algebraic group. It has $\mathrm{rank}(K) = 1$, so we get $\delta(G) = 0$ and might suspect the limit will be 0 for any choice of i. Consider any torsion-free congruence subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, which again acts on the upper half-plane. As we saw in Example 1.1, these subgroups are not cocompact, but the spaces $\Gamma \backslash \mathfrak{H}$ have finite volume. Then we have $H_i(\Gamma, M) \cong H_i(\Gamma \backslash \mathfrak{H}, \mathcal{M})$, where $\mathcal{M}$ is the local system induced by M. However, $\Gamma \backslash \mathfrak{H}$ deformation retracts to a finite graph, so $H_i(\Gamma, M)$ vanishes for $i > 1$, and when $i = 0, 1$ there is no torsion. Hence, for any congruence subgroup $\Gamma_n < \Gamma$ we have $\log |H_i(\Gamma_n, M)_{\text{tors}}| = 0$ for any i.

Very few cases of Conjecture 2.1 or 2.2 have been proven. However, Bergeron and Venkatesh showed that if $\mathbf{G} = \mathrm{SL}_2(\mathbb{C})$ or $\mathbf{G} = \mathrm{SL}_3(\mathbb{R})$, we can find subgroups and a module such that the limit converges to a positive number at degrees 1 and 2, respectively. In fact, these were corollaries of a more general theorem that they proved.

Theorem 2.1 ([3] Equation (1.4.2)): *Suppose that $\delta(\boldsymbol{G}) = 1$, then we can always find a sequence of cocompact arithmetic subgroups $\{\Gamma_n\}$ and a Γ-module M that is strongly acyclic (to be defined in Chapter 3) such that*

$$\lim_n \sum_j (-1)^{j+\frac{\dim(G/K)-1}{2}} \frac{\log |H_i(\Gamma_n, M)_{\text{tors}}|}{[\Gamma : \Gamma_n]} = c_{G,M} \operatorname{vol}(\Gamma \backslash G/K)$$

where $c_{G,M}$ is a positive constant.

Recently, Müller, Pffaf, and others have proven similar theorems, where a particular algebraic group is chosen, and cocompact arithmetic subgroups and modules are constructed so that the alternating sum of the logarithm of the torsion components, as in Theorem 2.1, is nonzero and related to the volume of the corresponding Riemannian symmetric space. Although these types of theorems do not isolate which cohomology group grows exponentially, they are impressive theorems in themselves and provide a large step towards proving Conjecture 2.1. The proofs of these theorems, including Theorem 2.1, hinge on Müller's work on analytic torsion, which shall be discussed in the next chapter.

3 Torsion

In this chapter, we introduce Reidemeister torsion and analytic torsion (also known as Ray-Singer torsion). Müller and Cheeger independently proved that under certain conditions, these two quantities are equal [4, 14] (see Theorem 3.1 below), which can be exploited to show results relating to Conjecture 2.1.

3.1 Reidemeister Torsion

In this section, the material and exposition follows sections 2.1 and 2.2 in [3]. Reidemeister torsion, sometimes called R-torsion, was first introduced by Reidemeister in 1935 to study three dimensional lens spaces, $L(p, q)$, which are the quotient spaces of S^3 modulo the relation $(z_1, z_2) \sim (e^{2\pi i/p} z_1, e^{2\pi i/q} z_2)$ for coprime integers p and q. These were the first known examples of 3-manifolds whose homeomorphism type is not determined by their homology and fundamental group. However, Reidemeister showed that three-dimensional lens spaces are classified up to homeomorphism by their fundamental group and Reidemeister torsion [17]. As we shall see, Reidemeister torsion is largely a topological invariant rather than an analytic or arithmetic invariant.

Consider a homomorphism of free finite rank $\mathbb{Z}$-modules $f : A_1 \to A_2$. First we fix an embeddings of A_1 and A_2 into $\mathbb{R}^n$, then find a $\mathbb{Z}$-basis $\{a_i \mid 1 \leq i \leq \text{rank}(A_1)\}$ for A_1, and similarly for A_2 [3]. Then we have the following equivalent definitions of the volume of a module:

$$\text{vol}(A_1) = \text{vol}(A_1 \otimes_{\mathbb{Z}} \mathbb{R}/A_1) = \sqrt{|\det(M)|} = ||a_1 \wedge \ldots \wedge a_n||$$

where M is the Gram matrix for our basis. Next, we define $\det'(f) = \prod_{i=1}^{\text{rank}(A_i)} \sigma_i$ where $\{\sigma_i\}$ is the set of nonzero singular values of f, that is, each $\sigma_i = \sqrt{\lambda_i}$ where λ_i is a positive eigenvalue of ff^T. With these definitions, we have

$$\text{vol}(A_1)\det'(f) = \text{vol}\left(\ker(f)\right)\text{vol}\left(\text{im}(f)\right) \qquad (*)$$

Suppose that we are given a cochain complex of free finite rank $\mathbb{Z}$-modules

$$0 \to A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} A^n \xrightarrow{d_n} 0$$

where $\text{vol}(A^i) = 1$ for each i. Notice that if each $H^i(A^*)$ is finite, $\text{im}(d_{i-1})$ is a sublattice of $\ker(d_i)$, and we have $|H^i(A^*)| = [\ker(d_i) : \text{im}(d_{i-1})] = \frac{\text{vol}(\text{im}(d_{i-1}))}{\text{vol}(\ker(d_i))}$ by $(*)$. In general, if the homology group is not finite, we have a "regulator" factor $R^i(A^*) := \text{vol}(H^i(A^*)_{\text{free}})$ which makes

$$\frac{\text{vol}(\ker(d_i))}{\text{vol}(\text{im}(d_{i-1}))} = |H^i(A^*)_{\text{tors}}|^{-1} R^i(A^*)$$

Furthermore, if the cochain complex A^* is exact, meaning that each homology group is trivial, then we call the cochain complex **acyclic**. With these definitions in place, we define the **Reidemeister torsion** of A^* be

$$\tau_A(d) := \prod_{i=0}^{n} \left(\det(d_i)\right)^{(-1)^i} = \prod_{i=0}^{n} |H^i(A^*)|_{\text{tors}}^{(-1)^{i+1}} R^i(A^*)^{(-1)^i}$$

where the second equality can be seen directly be substituting the equations we've written above. When the cochain complex is induced by a representation ρ, which will be our primary concern for the rest of the paper, we will denote the Reidemeister torsion as $\tau_A(\rho)$.

Example 3.1: Consider the complex of free $\mathbb{Z}$-modules

$$0 \to A^0 \xrightarrow{d_0} A^1 \to 0$$

where $A^0 \cong \mathbb{Z}^2$, $A^1 \cong \mathbb{Z}^2$, and $d_0 = \begin{pmatrix} k^2 & -k \\ -k & 1 \end{pmatrix}$. Then we imbed each copy of A^0 and A^1 into $\mathbb{R}^2$ using the standard basis $\left\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right\}$, and notice that the Gram matrix for each module is now the identity, so we've normalized in the sense that $\text{vol}(A^0) = \text{vol}(A^1) = 1$.

The rank of d_0 is 1, so already we can see that $H^0(A^*) \cong H^1(A^*) \cong \mathbb{Z}$, and hence the torsion components have cardinality 1. The image of d_0 is spanned by $\left\{\left(\begin{smallmatrix} -k \\ 1 \end{smallmatrix}\right)\right\}$, so we compute that $\text{vol}\left(\text{im}(d_0)\right) = \sqrt{\left\langle \left(\begin{smallmatrix} -k \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -k \\ 1 \end{smallmatrix}\right) \right\rangle} = \sqrt{k^2+1}$. Likewise, the kernel of d_0 is spanned by $\left\{\left(\begin{smallmatrix} 1 \\ k \end{smallmatrix}\right)\right\}$ so $\text{vol}\left(\ker(d_0)\right) = \sqrt{k^2+1}$. Next, we compute the eigenvalues of $d_0 d_0^T$ and find them to be 0 and $(k^2+1)^2$, and hence $\det'(d_0) = k^2+1$. Notice that this is consistent with our equations above. Finally, we can compute

$$R^0(A^*) = \text{vol}\left(\ker(d_0)\right) = \sqrt{k^2+1} \qquad \text{and} \qquad R^1(A^*) = \frac{1}{\text{vol}\left(\text{im}(d_0)\right)} = \frac{1}{\sqrt{k^2+1}}$$

and hence the Reidemeister torsion of our complex is $\tau_A(d) = k^2+1$.

3.2 Analytic Torsion

The material and exposition in this section follows [8]. First let us recall some definitions from differential geometry. Given a vector bundle $\pi : E \to X$ where the fibers are vector spaces V^n, we have local trivializations $\phi_A : A \times V^n \xrightarrow{\sim} \pi^{-1}(A)$ for small open subsets $A \subset X$. Given two open neighborhoods $A, B \subset X$ we have the composition function

$$\phi_A^{-1} \circ \phi_B : (A \cap B) \times V^n \to (A \cap B) \times V^n$$
$$\phi_A^{-1} \circ \phi_B : (x, v) \to (x, g_{AB}(x)v)$$

which defines the map $g_{AB} : A \cap B \to \mathrm{GL}_n(V)$ called the **transition function**. When E admits an open covering and local trivializations such that the transition functions are locally constant, we call E a **flat vector bundle**.

Now we will explain how analytic torsion is defined. Let X be an n-dimensional compact Riemannian manifold and $\rho : \pi_1(X) \to \mathrm{GL}_n(V)$ be a finite dimensional representation of the fundamental group of X. Given the universal cover $\tilde{X}$ of X, we can consider the trivial vector bundle $\pi : \tilde{X} \times V^n \to \tilde{X}$. This is flat because we can choose our transition functions to be the identity, which causes each transition function to map its domain to the identity matrix. Consider the action of $\pi_1(X)$ on $\tilde{X} \times V^n$ defined by

$$\gamma \cdot (\tilde{x}, v) = \big(\gamma\tilde{x},\ (\rho(\gamma))(v)\big)$$

where $\gamma\tilde{x}$ is the action of $\pi_1(X)$ on $\tilde{X}$ via deck transformations. Let $\sim$ be the equivalence relation formed by the action above, and recall that $X \cong \tilde{X}/\pi_1(X)$. By defining $E_\rho := (\tilde{X} \times V^n)/\sim$, we can construct the flat vector bundle associated to ρ

$$\pi_\rho : E_\rho \to X$$

In fact, there is a bijection between the flat vector bundles of rank n over X and the n-dimensional representations of the fundamental group $\pi_1(X)$ [8].

Because E_ρ is a flat vector bundle, there is a natural way of taking the exterior derivatives of differential forms on X with values in E_ρ. If $\{e_i\}$ is a basis for E_ρ, then we define the exterior derivative to be the usual exterior derivative acting component-wise. From the standard theory of differential geometry, we have the de Rham complex of differential forms with values in E_ρ

$$0 \to \Omega^0(X, E_\rho) \xrightarrow{d_0} \Omega^1(X, E_\rho) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \Omega^n(X, E_\rho) \to 0$$

Like the standard theory, we have the notion of the Hodge star operation. Using multi-index notation, consider a differential p-form $\omega = \sum_I f_I \mathbf{e}_i$. Then the **Hodge dual** is defined to be

$$*\omega = \sum_{I,J} \epsilon(IJ) f_I \mathbf{e}_J$$

where $I \cup J = \{1, \dots, n\}$, $I \cap J = \emptyset$, and $\epsilon(IJ)$ is the sign of the permutation of the sequence $(1, \dots, n)$ to (I, J). For example, if our 0-form takes values in $\mathbb{R}^n$, then $*f = f dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$. Similarly, in $\mathbb{R}^3$ the Hodge dual of a general 1-form is

$$*(f_1 dx_1 + f_2 dx_2 + f_3 dx_3) = f_1 dx_2 \wedge dx_3 - f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2$$

If we are considering differential forms on a smooth manifold with dimension n, then we define the **codifferential** map on a p-form to be

$$\delta = (-1)^{n(p+1)+1} * d*$$

Notice that the codifferential takes p-forms to $p-1$-forms. Our space of differential forms with compact support comes equipped with a natural inner product $\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta$, and one special property of the codiffernetial is the adjunction formula $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$ and $\langle \delta\alpha, \beta \rangle = \langle \alpha, d\beta \rangle$, which can be proved using Stokes' theorem. With this, we define the **Hodge Laplacian** on p-forms to be

$$\Delta = \delta d + d\delta$$

and to specify the Hodge Laplacian acting on p-forms, we write Δ_p. Notice that the adjunction formula above implies that Δ is self-adjoint in the sense that $\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$. Now let $\{\lambda_i\}_{i \in I}$ be the set of eigenvalues of Δ_p, counted with multiplicity. We define the ***p*-th zeta function** to be the complex function

$$\zeta_p(s) = \sum_{\lambda_i \geq 0} \lambda_i^{-s}$$

for the values of s for which the expression exists.

Now we are ready to define analytic torsion; given a closed Riemann manifold X of dimension n with vector bundle E_ρ as established above, we define the analytic torsion $T_X(\rho)$ to be

$$\ln\big(T_X(\rho)\big) = \frac{1}{2} \sum_{i \geq 0} (-1)^i i \frac{d}{ds}\Big|_{s=0} \zeta_i(s)$$

As it is defined above, it isn't clear whether or not the i-th zeta function converges. However, we have

Proposition 3.1 ([7] Lemma 1.10.1): *ζ_p is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > p/2$. Furthermore, it has an analytic continuation to a meromorphic function on $\mathbb{C}$ which is analytic at 0.*

Proof: For any self adjoint differential operator $\mathcal{O}$ we set

$$\zeta_{\mathcal{O}}(s) = \mathrm{Tr}(\mathcal{O}^{-s}) = \sum_i \lambda_i^{-s}$$

where λ_i are the eigenvalues of $\mathcal{O}$. With a slight abuse of notation, we will set $\zeta_p(s) := \zeta_{\Delta_p}(s)$, and from noting the identity

$$\int_0^\infty t^{s-1} e^{-\lambda t}\, dt = \lambda^{-s} \int_0^\infty (\lambda t)^{s-1} e^{-\lambda t}\, d(\lambda t) = \lambda^{-s} \Gamma(s)$$

we have

$$\begin{aligned}\int_0^\infty t^{s-1}\,\mathrm{Tr}(e^{-t\Delta_p})\,dt &= \int_0^\infty t^{s-1}\sum_i (e^{\lambda_i})^{-t}\,dt\\ &= \sum_i \int_0^\infty t^{s-1}e^{-t\lambda_i}\,dt\\ &= \sum_i \lambda_i^{-s}\Gamma(s)\\ &= \zeta_p(s)\Gamma(s)\end{aligned}$$

where we are using the fact that the eigenvalues of e^{Δ_p} are precisely e^{λ_i} where λ_i are the eigenvalues of Δ_p. Hence, we have the expression

$$\zeta_p(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}(e^{-t\Delta_p})\,dt$$

Further study of this function (as in [7]) shows that the expression above is holomorphic when $\mathrm{Re}(s) > \frac{p}{2}$, and has a meromorphic analytic continuation to $\mathbb{C}$ with possible simple poles at $(p-k)/2$ for $k = 0, 1, 2, \ldots$.

□

Example 3.2: Let's compute the analytic torsion of a circle $X = S^1$. To make things simpler, we take $S^1 \cong \mathbb{R}/\mathbb{Z}$ and consider the one-dimensional trivial representation of the fundamental group. This in turn induces the one-dimensional trivial vector bundle E_ρ. Since the circle is one dimensional, we get the sequence of differential forms

$$0 \to \Omega^0(S^1, E_\rho) \to \Omega^1(S^1, E_\rho) \to 0$$

and the formula for analytic torsion implies that we only need to compute $\zeta_1(s)$. By definition, we have $d(f) = f'\,dt$ and $d(f\,dt) = 0$. Likewise, the Hodge star sends $*(f) = f dt$ and $*(f\,dt) = f$. Hence, the codifferential operator acts by

$$\delta(f\,dt) = (-*d*)(f\,dt) = -f'$$

and the Laplacian is

$$\Delta_1(f\,dt) = (d\delta + \delta d)(f\,dt) = -f''\,dt$$

The eigenvectors of Δ_1 are spanned by $\{\cos(nt)dt, \sin(nt)dt, cdt\}$ where $n \in \mathbb{Z}_{>0}$ and $c \in \mathbb{C}$. These three families correspond to the eigenvalues $\{n^2, n^2, 0\}$, respectively. This implies that

$$\zeta_1(s) = 2\sum_{n\geq 1}(n^2)^{-s} = 2\zeta(2s)$$

where $\zeta(s)$ is the standard zeta function. Notice that $\zeta_1(s)$ is hence a meromorphic function with a single simple pole at $s = 1/2$, which agrees with our result from Proposition 3.1. Using the fact that $\lim_{s\to 0}\zeta'(s) = \frac{-\ln(2\pi)}{2}$, we can compute

$$\begin{aligned}\ln\big(T_{S^1}(\rho)\big) &= \frac{1}{2}\sum_{n\geq 0}(-1)^n n \frac{d}{ds}\Big|_{s=0}\zeta_n(s)\\ &= \frac{1}{2}(-1)\frac{d}{ds}\Big|_{s=0}\zeta_1(s)\\ &= \lim_{s\to 0} -2\zeta'(2s)\\ &= \ln(2\pi)\end{aligned}$$

so our analytic torsion is 2π.

3.3 R-Torsion from Arithmetic Groups

The material and exposition in this section follows section 2 of [15]. Suppose we want to compute the group cohomology $H^i(\Gamma, M)$ of a torsion-free arithmetic group $\Gamma < G$, where M is a free, finite rank $\mathbb{Z}$-module with a Γ-action. We start by constructing the Riemannian symmetric space $\tilde{X} = G/K$ as we saw in Chapter 1. Then we can construct the Eilenberg-MacLane space $X = \Gamma\backslash\tilde{X}$, which is a connected smooth manifold of dimension n with $\pi_1(X) = \Gamma$. Since X is smooth, we can construct a triangulation L, which lifts to a triangulation $\tilde{L}$ of $\tilde{X}$. We can work with the free abelian groups $C_i(L,\mathbb{Z})$, $C_i(\tilde{L},\mathbb{Z})$, and $C^i(\tilde{L},\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}\big(C_i(\tilde{L},\mathbb{Z}),\mathbb{Z}\big)$. However, we want to work with M coefficients instead of $\mathbb{Z}$.

From the homeomorphism $X \simeq \Gamma\backslash\tilde{X}$, we know that the i-cells of L form a basis for $C_i(\tilde{L},\mathbb{Z})$ over $\mathbb{Z}[\Gamma]$. Viewing M as a $\mathbb{Z}[\Gamma]$-module, we can define

$$C^i(L,M) := C^i(\tilde{L},\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} M \cong \mathrm{Hom}_{\mathbb{Z}[\Gamma]}\big(C_i(\tilde{L},\mathbb{Z}), M\big)$$

This cochain complex gives us an isomorphism of homology groups

$$H^i(L,M) \cong H^i(\Gamma,M)$$

for all i. We can consider the module $V := M \otimes_{\mathbb{Z}} \mathbb{C}$ and the cochains

$$C^i(L,V) := C^i(\tilde{L},\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} V \cong C^i(L,M) \otimes_{\mathbb{Z}} \mathbb{C}$$

We now define a representation $\rho : \Gamma \to \mathrm{GL}(M)$ to be **unimodular** if $|\det(\gamma)| = 1$ for all $\gamma \in \Gamma$. Since M is a $\mathbb{Z}$-lattice, for any representation ρ on M, the determinant of the image of any $\gamma \in \Gamma$ must be an integer. However, since ρ is a homomorphism, we must have $|\det(\gamma)| = 1$. Furthermore, we can regard ρ as a unimodular representation of Γ on V.

As in the previous section, let $E := \tilde{X} \times_\rho V$ be the flat vector bundle over X associated to ρ. Then we can compute the de Rham cohomology groups $H^i(X,E)$ of complex E-valued differential forms on X. From the de Rham isomorphism, we get that $H^i(L,V) \cong H^i(X,E)$ for all i. Notice that so far, we have not imposed any restrictions on our module M, besides it being a free finite rank $\mathbb{Z}$-module. Now, we call M and ρ **acyclic** if the sequence

$$\ldots \longrightarrow C^i(L,M)\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{d_i} C^{i+1}(L,M)\otimes_{\mathbb{Z}}\mathbb{C} \longrightarrow \ldots$$

is acyclic. Recall that this means the cohomology groups $H^i(L,V) \cong H^i(X,E) \cong 0$ are trivial for all i. As a consequence, we can show that each $H^i(\Gamma, M)$ must be finite.

By the Universal Coefficient theorem for cohomology, we know that

$$0 \cong H^i(L,V) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(L,\mathbb{Z}),V) \oplus \mathrm{Hom}_{\mathbb{Z}}(H_i(L,\mathbb{Z}),V)$$

where we are considering V as a $\mathbb{Z}$-module. Notice that $\mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(L,\mathbb{Z}),V) \cong 0$ since V is an injective $\mathbb{Z}$-module. This implies that $H_i(L,\mathbb{Z})$ must be finite for each i, or else we could find a nontrivial homomorphism from it to V.

Now suppose that M has rank n, then the Universal Coefficient theorem for homology implies that

$$\begin{aligned} H_i(L,M) &\cong \big(H_i(L,\mathbb{Z}) \otimes_{\mathbb{Z}} M\big) \oplus \mathrm{Tor}_1(H_{i-1}(L,\mathbb{Z}),M) \\ &\cong \prod^n \big(H_i(L,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\big) \oplus 0 \\ &\cong H_i(L,\mathbb{Z})^n \end{aligned}$$

where we have used the fact that M is a flat $\mathbb{Z}$-module (because it is free) so the Tor term is 0. This means that $H_i(L,M)$ must be finite, and by Poincaré duality for torsion, we get that

$$H_i(\Gamma,M) \cong H_i(L,M) \cong H^{n-i-1}(L,M) \cong H^{n-i-1}(\Gamma,M)$$

and hence $H^i(\Gamma,M)$ is finite for all i.

The following proposition gives the connection between Reidemeister torsion and the torsion in the group cohomology of arithmetic groups.

Proposition 3.2 ([15] Proposition 2.1): *Given the setting above, where $\rho : \Gamma \to \mathrm{GL}(M)$ is an acyclic representation of Γ on a free finite rank $\mathbb{Z}$-module,*

$$\ln\big(\tau_X(\rho)\big) = \sum_{i=0}^{n} (-1)^{i+1} \ln\big(|H^i(\Gamma,M)|\big)$$

<u>Proof:</u> The majority of the proof has already been finished above. Since the cohomology groups $H^i(\Gamma,M)$ are finite, we compute the Reidemeister torsion of the cochain complex

$$\ldots \longrightarrow C^i(\tilde{L},\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} M \xrightarrow{d_i} C^{i+1}(\tilde{L},\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} M \longrightarrow \ldots$$

and take the logarithm to get

$$\ln\big(\tau_X(\rho)\big) = \sum_{i=0}^{n} (-1)^i \ln\big(\det(d_i)\big) = \sum_{i=0}^{n} (-1)^{i+1} \ln\big(|H^i(\Gamma,M)|\big)$$

□

When ρ is acyclic, Proposition 3.2 provides a bridge between the group cohomology and R-torsion, which is what Bergeron and Venkatesh exploited to prove Theorem 2.1. The problem is that Reidemeister torsion is kind of clunky to work out explicitly—there isn't an easy way to see how it grows. This is where the connection between analytic torsion and Reidemeister torsion comes into play. Independently of each other, Cheeger and Müller proved

Theorem 3.1 ([4, 12]): *When ρ is a unitary representation, $T_X(\rho) = \tau_X(\rho)$.*

where a unitary representation $\rho : \Gamma \to \mathrm{GL}(M)$ is one where $\rho(\gamma)$ is unitary for every $\gamma \in \Gamma$. Müller later generalized this result by showing

Theorem 3.2 ([13] Theorem 1.1): *When ρ is a unimodular representation, $T_X(\rho) = \tau_X(\rho)$.*

Combined with Proposition 2.1, this result implies that we are able to study the group cohomology of arithmetic subgroups with analytic means by examining analytic torsion. This is exactly what Bergeron and Venkatesh did to prove Theorem 2.1, though they imposed the stricter requirement that the Γ-module M be **strongly acyclic**, meaning the eigenvalues of every Δ_p for any choice of $\Gamma_n < \Gamma$ are uniformly bounded away from 0. At first, it may be surprising that strongly acyclic modules exist, or even that acyclic modules exist. However, part of Theorem 2.1 was showing that we can construct such modules for any algebraic group with deficiency equal to 1. For example, when considering the algebraic group $\mathrm{SL}_2(\mathbb{C})$ (which has deficiency 1), the $\mathrm{SL}_2(\mathbb{Z}[i])$-module $\mathrm{Sym}^p(\mathbb{Z}[i]^2) \otimes_{\mathbb{Z}} \overline{\mathrm{Sym}^q(\mathbb{Z}[i]^2)}$ is acyclic if and only if $p \neq q$ [3]. However, Theorem 2.1 and Conjecture 2.1 do not apply becuase the lattice $\mathrm{SL}_2(\mathbb{Z}[i])$ is not cocompact.

4 From Analytic Torsion to the B-V Conjecture

4.1 From Subgroups to Modules

When Bergeron and Venkatesh first proposed their conjecture, they fixed a finite rank free $\mathbb{Z}$-module M, and studied the growth of the torsion in $H_i(\Gamma_n, M)$ when passing through a decreasing sequence of cocompact congruence subgroups $\ldots \subset \Gamma_n \subset \ldots \subset \Gamma$ where $\cap_n \Gamma_n = \{1\}$. In work done by Müller and Pfaff, they fix the arithmetic group Γ and study the torsion of $H_i(\Gamma, M_n)$ for a sequence of Γ-modules $\{M_n\}$. In both cases, the theorems they prove make use of modules with unimodular representations so as to utilize Theorem 3.2 [13] relating the torsion in the group cohomology with the analytic torsion. Here, we will highlight the correspondence between these two approaches.

Proposition 4.1: *Let H be a finite index subgroup of G. If $\rho : H \to \mathrm{GL}(V)$ is a unimodular representation, then the induced representation $\tilde{\rho} : G \to \mathrm{GL}\left(\mathrm{Ind}_H^G(V)\right)$ is also unimodular.*

<u>Proof:</u> Let $[G : H] = n$ and pick coset representatives $g_1, \ldots, g_n$. By definition, our module is defined by $\mathrm{CoInd}_H^G(V) = \{f : G \to V \mid f(hg) = \rho(h)\big(f(g)\big)\}$ and the action of G on it is defined by

$$\begin{aligned} \tilde{\rho} &: G \times \mathrm{CoInd}_H^G(V) \to \mathrm{CoInd}_H^G(V) \\ \tilde{\rho} &: \big(g^*, f(g)\big) \mapsto f(gg^*) \end{aligned}$$

Since any $g \in G$ can be written as $g = g_i h_i$ for one of our coset representatives g_i and $h_i \in H$, a map $f \in \mathrm{Ind}_H^G(V)$ is completely determined by where it sends each coset rep. If we fix a basis

$\{v_1, \ldots, v_m\}$ for V, the space $\mathrm{Ind}_H^G(V)$ is thus spanned by maps $f_{i,j}$ which send $f_{i,j}(g_i) = v_j$, and which are zero on all other coset reps. From this, we confirm that $Ind_H^G(V)$ has dimension nm.

If we focus on how our action affects these basis members, we see that

$$\tilde{\rho} : \big(g_i, f_{j,k}\big) = f_{l,k} \qquad \text{where } g_j g_i \in g_l H$$

Hence if we let $\rho(y) = 0$ for all $y \notin H$, then we can express

$$\tilde{\rho}(x) = \begin{pmatrix} \rho(g_1^{-1} x g_1) & \ldots & \rho(g_1^{-1} x g_n) \\ \vdots & \ddots & \vdots \\ \rho(g_n^{-1} x g_1) & \ldots & \rho(g_n^{-1} x g_n) \end{pmatrix}$$

Notice that each $\rho(g_i^{-1} x g_j)$ is an m-by-m matrix. Furthermore, each xg_j belongs to a distinct coset of G/H, and hence there is exactly one coset rep g_i such that $g_i^{-1} x g_j \in H$ (which makes $\rho(g_i^{-1} x g_j) \neq 0$). That is, there is exactly one non-zero block matrix in each column. By the same reasoning, there is exactly one non-zero block matrix in each row.

By rearranging the rows of $\tilde{\rho}(x)$, we can get a diagonal block matrix which has the same determinant as $\tilde{\rho}(x)$, up to sign. However, notice that each $\rho(g_i^{-1} x g_j)$ has determinant 1 because ρ is unimodular. Since the determinant of a block diagonal matrix is the product of the determinants of its blocks, we get $|\tilde{\rho}(x)| = 1$. Finally, since H has finite index in G, the coinduced and induced representations coincide, and hence our induced representation is unimodular.

□

Let $\Gamma < G$ be an arithmetic subgroup of a semi-simple connected algebraic group over $\mathbb{Q}$. Given a finite rank free $\mathbb{Z}$-module M, suppose there exists a decreasing sequence of congruence subgroups $\{\Gamma_n\}_{n=1}^{\infty}$ such that $|H^i(\Gamma_n, M)|_{tors}$ grows exponentially. As we explained in Chapter 3, M must be unimodular, and since each Γ_n is of finite index in Γ, Shapiro's Lemma implies that

$$H^i(\Gamma_n, M) \cong H^i\big(\Gamma, \mathrm{Ind}_{\Gamma_n}^{\Gamma}(M)\big)$$

By Proposition 4.1, each induced representation is unimodular, and hence a sequence of subgroups yields a sequence of modules with unimodular representations which cause the cohomology to grow exponentially.

With all of this in place, it makes sense to focus on varying the modules instead of the subgroups, because we can still utilize Theorem 3.2. In fact, this is largely what has been done in the literature so far. For instance Müller and Pfaff [15] constructed certain torsion-free cocompact arithmetic groups Γ of $SO^0(p,q)$ for p, q odd, and showed that there exists modules M_m for each $m \in \mathbb{N}$ such that

$$\lim_{m\to\infty} \sum_i (-1)^i \log |H^i(\Gamma, M_m)_{\mathrm{tors}}| = -C_{p,q} \operatorname{vol}(\Gamma\backslash\tilde{X}) m \operatorname{rank}_{\mathbb{Z}}(M_m) + O(\operatorname{rank}_{\mathbb{Z}}(M_m))$$

Notice that this implies there exists at least one i such that $|H^i(\Gamma, M_m)_{\mathrm{tors}}|$ grows exponentially, however it is still conjectural that the cohomological degree which does so is $i = \frac{\dim(\tilde{X})+1}{2}$. In the same paper, Müller and Pfaff proved a similar result for the algebraic group $\mathrm{SL}_3(\mathbb{R})$ and certain cocompact subgroups Γ.

4.2 Algebraic Groups with $\delta \neq 1$

Most of the research concerning Conjecture 2.1 that has been published has focused on cases when the deficiency is equal to one, which includes $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_4(\mathbb{R})$, and $\mathrm{SO}(p,q)$ where pq is odd. However, for Conjecture 2.1 to be complete, it still needs to be shown that little torsion appears in cases when the deficiency isn't equal to one. Towards this goal, there have been several results, for instance,

Theorem 4.1 ([16] Theorem 2.3): *Suppose X is an even-dimensional oriented compact manifold without boundary and ρ a finite dimensional representation of $\pi_1(X)$, then $T_X(\rho) = 1$.*

If G/K has even dimension (for instance $G = \mathrm{SU}(p,q)$ or $G = \mathrm{SO}(p,q)$ with pq even, both of which are examples when $\delta(\mathbf{G}) = 0$), then any cocompact lattice $\Gamma < G$ will share the property above of having trivial analytic torsion. This can be seen as a manifestation of the Poincaré duality; suppose X has dimension n and M is a finite rank free $\mathbb{Z}$-module, then there are isomorphisms which result in $|H_i(\Gamma, M)_{\mathrm{tors}}| = |H_{n-i-1}(\Gamma, M)_{\mathrm{tors}}|$ and $R_i(A^*)R_{n-i}(A^*) = 1$ (where A is the cochain complex derived in Section 3.3). Since the representation associated to M will be unimodular, Theorem 3.2 gives us

$$T_X(\rho) = \tau_x(\rho) = \prod_{i=0}^{n} |H^i(A^*)|_{\mathrm{tors}}^{(-1)^{i+1}} R^i(A^*)^{(-1)^i} = R^{i/2}(A^*)$$

which can be shown to equal 1.

Likewise, we have the following theorem due to Müller and Pfaff

Theorem 4.2 ([14] Theorem 1.1): Suppose $G = \mathbf{G}(\mathbb{R})$ is semi-simple with finite center and of non-compact type. If $X = G/K$ is even dimensional or $\delta(\mathbf{G}) \neq 1$, then $T_X(\tau) = 1$ for all finite-dimensional representations τ of G.

Suppose that X is odd dimensional and $\delta(\mathbf{G}) \neq 1$ (for instance $\mathrm{SL}_n(\mathbb{R})$ for certain $n \geq 5$). If ρ is an acyclic representation defined by the restriction of a finite dimensional representation of G to a cocompact Γ, then Proposition 3.2 and the theorem above imply

$$0 = \ln(T_X(\rho)) = \ln(\tau_X(\rho)) = (-1)^{\frac{n+1}{2}} |H^{\frac{n-1}{2}}(\Gamma, M)|$$

However, there is no reason to assume that many representations of Γ would be acyclic (recall that acyclic representation result in each cohomology group $H^i(\Gamma, M)$ being finite), or that ρ should be a restricted representation of G.

5 Appendix A

Throughout this paper, we have been interested in how torsion behaves in the homology of a sequence of subgroups of an algebraic group. The counterpart to this question is how the rank of these homology groups behaves asymptotically, which slightly more is known about. For instance, consider a lattice Γ (not necessarily cocompact) in a semisimple Lie group G, and the Riemannian space $Y = \Gamma\backslash G/K$. Let $\Omega^p(Y)$ denote the C^∞ p-forms on Y and $L^2(Y)$ be the completion of

$\Omega^p(Y)$ with respect to the L^2-metric. Setting $\Omega^p_{(2)}(Y) := \Omega^i(Y) \cap L^2(Y)$, we can consider the cochain complex

$$0 \to \Omega^0_{(2)}(Y) \xrightarrow{d} \Omega^1_{(2)}(Y) \xrightarrow{d} \Omega^2_{(2)}(Y) \xrightarrow{d} \dots$$

where d is the exterior differential with domain $\{\alpha \in \Omega^i_{(2)}(Y) \,|\, d\alpha \in L^2(Y)\}$. The cohomology of this complex is called the L^2**-cohomology**, and Wolfgang Lück showed that given a decreasing sequence of subgroups $\Gamma > \Gamma_1 > \Gamma_2 > \dots$ with trivial intersection, the quotient

$$\lim_{n\to\infty} \frac{\dim H_i(\Gamma_n, \mathbb{C})}{[\Gamma : \Gamma_n]}$$

converges to the ith L^2-Betti number of Y [10]. Furthermore, it can be shown that this implies the limit is nonzero only when the deficiency of G is zero.

As an explicit example, consider again $G = \mathrm{SL}_2(\mathbb{R})$ (which we showed in Chapter 2 has deficiency zero) and the lattice $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let $K = SO(2)$ be the maximal compact subgroup of G, and $X = G/K$ be the resulting symmetric space. Consider a sequence of congruence subgroups $\Gamma > \Gamma_1 > \Gamma_2 > \dots$ with trivial intersection, then since we are considering coefficients in a field, we can identify the homology groups

$$H_i(\Gamma_n, \mathbb{C}) \cong H_i(\Gamma_n \backslash X, \mathbb{C})$$

for all i and n.

Recall that the principal congruence subgroup of level N, denoted $\Gamma(N)$, is defined as the kernel of the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ which reduces each entry modulo N. The properties of principal congruence subgroups have been well-studied. For instance [19], we know that the number of cusps for $\Gamma(N)$ is equal to $\frac{N^2}{2}\prod_{p|N}(1-\frac{1}{p^2})$ and the genus of $X(N)$, the compactification of $\Gamma(N)\backslash X$, is $1 + \frac{N^2(N-6)}{12}\prod_{p|N}(1-\frac{1}{p^2})$. One can show using the identification space of a genus g surface that the fundamental group of a genus g surface with k punctures is the free group generated by $2g + n - 1$ elements. Therefore, since the non-compact space $Y(N) = \Gamma(N)\backslash X$ will have the same genus as $X(N)$ and a puncture for every cusp, we can compute

$$H_i\big(\Gamma(N), \mathbb{C}\big) \cong \begin{cases} \mathbb{C} & i = 0 \\ \mathbb{C}^{\phi(N)} & i = 1 \\ 0 & i \geq 2 \end{cases} \qquad \text{where} \qquad \phi(N) = \frac{N^2(N-3)}{6}\prod_{p|N}\big(1 - \frac{1}{p^2}\big)$$

Because there are only finitely many congruence subgroups of any given level, the level of the nth congruence subgroup in our sequence must go to infinity as n increases. Also notice that our sequence of subgroups implies that the lower levels must divide any higher levels. That is, if N_1 is the level of Γ_n and N_2 is the level of Γ_m where $m > n$, then $N_1 | N_2$. Our goal will be to use the homology of principal congruence subgroups, which are easy to compute explicitly, to compute the homology of each Γ_n by using spectral sequences.

Pick any subgroup Γ_n in our sequence, and suppose it has level N. The Lyndon-Hochschild-Serre spectral sequence is a spectral sequence of homological type

$$H_p\big(\Gamma/\Gamma(N),\, H_q(\Gamma(N), \mathbb{C})\big) \Longrightarrow H_{p+q}(\Gamma, \mathbb{C})$$

Then the E^2 page will take the form

	$\vdots$	$\vdots$	$\vdots$	
2	$H_0\big(\Gamma/\Gamma(N), H_2(\Gamma(N),\mathbb{C})\big)$	$H_1\big(\Gamma/\Gamma(N), H_2(\Gamma(N),\mathbb{C})\big)$	$H_2\big(\Gamma/\Gamma(N), H_2(\Gamma(N),\mathbb{C})\big)$	$\dots$
1	$H_0\big(\Gamma/\Gamma(N), H_1(\Gamma(N),\mathbb{C})\big)$	$H_1\big(\Gamma/\Gamma(N), H_1(\Gamma(N),\mathbb{C})\big)$	$H_2\big(\Gamma/\Gamma(N), H_1(\Gamma(N),\mathbb{C})\big)$	$\dots$
0	$H_0\big(\Gamma/\Gamma(N), H_0(\Gamma(N),\mathbb{C})\big)$	$H_1\big(\Gamma/\Gamma(N), H_0(\Gamma(N),\mathbb{C})\big)$	$H_2\big(\Gamma/\Gamma(N), H_0(\Gamma(N),\mathbb{C})\big)$	$\dots$
	0	1	2	3

From the Third isomorphism Theorem, $\Gamma/\Gamma(N)$ is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, and in particular is a finite group. From the study of homological algebra, we know that the integral homology (for $i > 0$) of a finite group is all torsion [20], which means that taking $\mathbb{C}$ coefficients will kill the group, so we have $H_i(\Gamma/\Gamma(N),\mathbb{C}) \cong 0$ for all $i > 0$. Our E^2 page then becomes

	$\vdots$	$\vdots$	$\vdots$	
2	0	0	0	$\dots$
1	$\mathbb{C}^{\phi(N)}$	0	0	$\dots$
0	$\mathbb{C}$	0	0	$\dots$
	0	1	2	3

and hence $\dim H_1(\Gamma_n,\mathbb{C}) = \phi(N)$.

Now suppose that the subgroups in our sequence $\{\Gamma_n\}_{n\in\mathbb{Z}}$ tend to get sufficiently close to the principal congruence subgroup that they contain. That is, if $[\Gamma : \Gamma_n] \sim [\Gamma : \Gamma(N)]$ as n (and hence N) go to infinity, then since $[\Gamma : \Gamma(N)] = N^3 \prod_{p|N}\left(1 - \frac{1}{p^2}\right)$, we get the non-zero converging limit

$$\lim_{n\to\infty} \frac{\dim H_1(\Gamma_n,\mathbb{C})}{[\Gamma : \Gamma_n]} = \frac{1}{6}$$

Acknowledgments

I feel so grateful to have worked with my advisor Professor Mark McConnell during my time at Princeton. I can't thank you enough for all of the support and guidance that you've given me over the years. Throughout your junior seminar, my junior paper, our summer research, and this thesis, you have been an incredible mentor and one of my greatest inspirations.

I want to thank my professors and peers in the mathematics department. You all have given me a better understanding and appreciation for math. I especially want to thank Professor Francesc Castella, who taught me the theory of elliptic curves and exposed me to mathematical research, Professor Sophie Morel, who introduced me to the vast field of algebraic geometry, and Professor Christopher Skinner, for being the second reader of this thesis.

Finally, I want to thank my friends and my family for their support. Your love has kept me sane and happy.

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