An Exposition of the First and Second Theorems in GAGA

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May 14, 2019

Abstract

In this paper, we will present a short exposition of the first and second major theorems in Jean-Pierre Serre's *Géométrie Algébrique et Géométrie Analytique*, as well as the background needed to understand it. We will assume that the reader is familiar with basic algebraic geometry, such as that found in Qing Liu's *Algebraic Geometry and Arithmetic Curves.*

Contents

1 Complex Analytic Spaces

First we need to introduce the concept of a complex analytic space, which generalizes complex manifolds to allow for singularities. Throughout this section let $U \subset \mathbb{C}^n$, and $\mathcal{H}_U$ be the sheaf of holomorphic functions on U. We define an *affine complex analytic space* to be a locally ringed space $(X, \mathcal{H}_X)$ such that

$$X = V(f_1, \dots, f_k) = \{x \in U \mid f_1(x) = \dots = f_k(x) = 0\}$$

for some holomorphic functions f_i on U. We define its coherent structure sheaf $\mathcal{H}_X$ as follows; let $\mathcal{I}$ be the coherent sheaf of ideals associated to the functions f_i, and then using the inclusion map $\iota : X \to U$, we set $\mathcal{H}_X := \iota^{-1}\big(\mathcal{H}_U/\mathcal{I}\big)$ to be the inverse image. Recall that on the stalks, we have the nice property $\mathcal{H}_{X,x} = \mathcal{H}_{U,x}/(f_1, \dots, f_k)$ for all $x \in X$, and we call $\mathcal{H}_{X,x}$ the *ring of germs of holomorphic functions.*

Using this, we define more generally a *complex analytic space* to be a locally ringed, separated space $(X, \mathcal{H}_X)$ with an open cover $\{V_i\}$ of X such that each $(V_i, \mathcal{H}_X|_{V_i})$ is an affine complex analytic space. Given two analytic sets $U \subset \mathbb{C}^r$ and $V \subset \mathbb{C}^{r'}$, a map $\phi : U \to V$ is an *analytic holomorphism* if it is continuous, and for every $f \in \mathcal{H}_{V,\phi(x)}$, we have $f \circ \phi \in \mathcal{H}_{U,x}$. Furthermore, we say ϕ is an *analytic isomorphism* if it is holomorphic and has a holomorphic inverse ϕ^{-1}. Adding one more to our list of definitions, we define an *analytic sheaf* $\mathcal{F}$ on a complex analytic space X to be a sheaf of modules over the sheaf of rings $\mathcal{H}_X$

Now that our definitions are out of the way, we are ready to start proving results.

Propostion 1: The sheaf $\mathcal{H}_X$ is a coherent sheaf of rings.

Proof: Since our proposition is local, we may reduce to the case where X is a closed subset of an open $U \subset \mathbb{C}^n$. By the Oka coherence theorem, we know that the sheaf of holomorphic functions over a complex manifold is coherent, which implies that $\mathcal{H}_U$ is a coherent sheaf of rings. This is explained in [5] (who directs the reader to [3] for a proof which is too long to write out here). Then since $\mathcal{H}_U$ is a coherent sheaf of rings and $\mathcal{I}$ is a coherent sheaf of ideals, Theorem 3 of Section 16 in [5] implies that $\mathcal{H}_X$ is a coherent sheaf of rings.

$\square$

Note that we know the structure sheaf of a ringed topological space is always quasi-coherent, but it's not necessarily coherent (viewed as a module over itself). Indeed, the main obstruction keeping the structure sheaf from being coherent is when its stalks aren't noetherian. This can be seen in part from Proposition 1.11 of [4], which implies that the structure sheaf of a locally noetherian scheme is coherent. Coinciding with this and Proposition 1, we can show that the structure sheaf for complex analytic spaces has Noetherian stalks.

To see this, first consider the case when $X = \mathbb{C}^n$. Then for any $x \in \mathbb{C}^n$, the stalk $\mathcal{H}_{\mathbb{C}^n,x} = \mathbb{C}\{x_1, \dots, x_n\}$ is the ring of formal power series that converge in a neighborhood of x. To show that this is noetherian we will proceed by induction on n; for the base case there is nothing to show since any field is noetherian. Now suppose we know $\mathbb{C}\{x_1, \dots, x_{n-1}\}$ is noetherian. By the Weierstrass preparation theorem, we can write any element in $\mathbb{C}\{x_1, \dots, x_n\}$ as a unit times an element in $\mathbb{C}\{x_1, \dots, x_{n-1}\}[x_n]$, which means that by Hilbert's basis theorem and our inductive hypothesis, we can conclude our claim. In general, the ring of germs of holomorphic functions on a complex analytic space takes the form $\mathbb{C}\{x_1, \dots, x_n\}/I$ for some ideal I. This is still noetherian since quotients of noetherian rings are also noetherian, so we can conclude that $\mathcal{H}_X$ is noetherian at its stalks.

2 Analytification

Given some scheme X of finite type over $\mathbb{C}$, we would like to associate to it a complex analytic space, which we will denote by X^{an}. In the affine case, this is clear; we associate the scheme $X = \text{Spec}\left(\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_k)\right) = \text{Spec}\left(\mathbb{C}[x_1, \dots, x_n]/I\right)$ with the space $X^{\text{an}} = V(f_1, \dots, f_k)$, which inherits a topology induced from the standard topology on $\mathbb{C}^n$, as well as the structure sheaf described above. As a set, X^{an} is the maximal ideals of X, which correspond to the $\mathbb{C}$-rational points of our scheme, commonly denoted as $X(\mathbb{C})$. As for the sheaf, we define $\mathcal{H}_{X^{\text{an}}}$ to be the sheaf generated on each stalk by $\mathcal{H}_{X,x}/I\mathcal{H}_{X,x}$.

Intuitively, the analytification of a scheme is how we typically “visualize” the scheme, and we'd like to consider it as if it were a manifold except possibly with singularities. In general, if X is a scheme, we take an open affine cover, glue together the corresponding affine complex analytic space, an obtain a complex analytic space, $(X^{\text{an}}, \mathcal{H}_{X^{\text{an}}})$, which we call the *analytification of* X.

It's important to note that the analytic topology on X^{an} is finer than the Zariski topology on X. This is because a subset of $\mathbb{C}^n$ is closed under the Zariski topology if it is the zero set of some finite number of polynomials, and these sets are also closed under the standard topology.

We also want an analytificiation of sheaves so that we can pullback sheaves on X to get sheaves on X^{an}. We will do so in the following way; consider the continuous inclusion map $\phi : X^{\text{an}} \to X$ and a sheaf $\mathcal{F}$ on X. Using this, we set $\mathcal{F}^{\text{an}} := \phi^{-1}\mathcal{F} \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{H}_{X^{\text{an}}}$ which is a sheaf of $\mathcal{H}_{X^{\text{an}}}$-modules, and therefore an analytic sheaf.

Propostion 2: The functor which sends $\mathcal{F}$ to $\mathcal{F}^{\text{an}}$ is exact. Also, if $\mathcal{F}$ is a coherent algebraic sheaf, then $\mathcal{F}^{\text{an}}$ is a coherent analytic sheaf.

Proof: For the first part, suppose that $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on X. By Proposition 2.2.18 of [4], we know that a sequence of sheaves is exact if and only if it is exact at all of the stalks. In our case, the stalks of each $\mathcal{F}_{i,x}$ are equal to the stalks $(\phi^{-1}\mathcal{F}_i)_x$ for any $x \in X^{\text{an}}$. Therefore applying the proposition above twice, we get that $0 \to \phi^{-1}\mathcal{F}_1 \to \phi^{-1}\mathcal{F}_2 \to \phi^{-1}\mathcal{F}_3 \to 0$ is exact.

It can be shown (Corollary 3 of [6]) that $(\mathcal{O}_{X,x}, \mathcal{H}_{X^{\text{an}},x})$ is a flat couple, but we will admit the proof here because it relies on Propositions 3, 22, 23, 27, and 28 in [6]. With this principle, we get that

$$\phi^{-1}\mathcal{F}_1 \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{H}_{X^{\text{an}}} \longrightarrow \phi^{-1}\mathcal{F}_2 \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{H}_{X^{\text{an}}} \longrightarrow \phi^{-1}\mathcal{F}_3 \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{H}_{X^{\text{an}}}$$

is exact from our reasoning above, and we can conclude our first claim.

Now to show the second part, if we let $\mathcal{O}_X^n$ denote the direct sum of n copies of $\mathcal{O}_X$, then since $\mathcal{F}$ is coherent, locally there exist integers p and q such that $\mathcal{O}_X^q \to \mathcal{O}_X^p \to \mathcal{F} \to 0$ is an exact sequence. Note that by definition, $\mathcal{O}_X^{\text{an}} = \mathcal{H}_{X^{\text{an}}}$, and the first part of this proposition implies that locally we have an exact sequence $\mathcal{H}_{X^{\text{an}}}^q \to \mathcal{H}_{X^{\text{an}}}^p \to \mathcal{F}^{\text{an}} \to 0$, which shows that our analytic sheaf is quasi-coherent.

If we denote the map from $\mathcal{H}_{X^{\text{an}}}^q$ to $\mathcal{H}_{X^{\text{an}}}^p$ by α, then we get a short exact sequence $0 \to \text{im}(\alpha) \to \mathcal{H}_{X^{\text{an}}}^p \to \text{coker}(\alpha) \to 0$. Proposition 1 implies that $\mathcal{H}_{X^{\text{an}}}^p$ is coherent, and we know that the image of a morphism between coherent sheaves is also coherent. Finally, noting that $\text{coker}(\alpha) \cong \mathcal{F}^{\text{an}}$, we can evoke Theorem 1 in Section 13 of [5]; this theorem states that in a short exact sequence of sheaves, if two sheaves are coherent, then so is the third, so we can conclude that $\mathcal{F}^{\text{an}}$ is coherent.

$\square$

Before we dive into proving the first major theorem of GAGA, we need one more tool which will allow us to simplify the situation later. Suppose that X is a subvariety, closed under the Zariski topology, of an algebraic variety Y. If $\mathcal{F}$ is a coherent algebraic functor on X, then we can consider its extension by the zero functor by setting $\mathcal{F}^Y$ to be the functor which is $\mathcal{F}$ on X and 0 on $Y \setminus X$. Likewise, we can consider the analytic sheaf $(\mathcal{F}^{\text{an}})^Y$ to be the extension of $\mathcal{F}^{\text{an}}$ to a functor which is 0 on $Y^{\text{an}} \smallsetminus X^{\text{an}}$.

Propostion 3: Using the notation from directly above, we have a canonical isomorphism between $(\mathcal{F}^{\text{an}})^Y$ and $(\mathcal{F}^Y)^{\text{an}}$.

Proof: By definition, the two sheaves are zero outside of X^{an}, so it suffices to show that their restrictions to X^{an} are isomorphic. For any $x \in X^{\text{an}}$, we have

$$(\mathcal{F}^{\text{an}})^Y_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_{X,x} \quad \text{and} \quad (\mathcal{F}^Y)^{\text{an}}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{Y,x}} \mathcal{H}_{Y,x}$$

Since $\mathcal{H}_{X,x} = \mathcal{H}_{Y,x}/I$ for an ideal I, we also have $\mathcal{O}_{X,x} = \mathcal{O}_{Y,x}/I$ so there is a canonical isomorphism $\mathcal{H}_{X,x} \cong \mathcal{H}_{Y,x} \otimes_{\mathcal{O}_{Y,x}} \mathcal{O}_{X,x}$. By the associativity and commutative of the tensor product, we have

$$\begin{aligned}
(\mathcal{F}^{\text{an}})^Y_x &\cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_{Y,x} \otimes_{\mathcal{O}_{Y,x}} \mathcal{O}_{X,x} \\
&\cong (\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{Y,x}} \mathcal{H}_{Y,x} \\
&\cong (\mathcal{F}^Y)^{\text{an}}_x
\end{aligned}$$

We've now shown that $(\mathcal{F}^{\text{an}})^Y$ and $(\mathcal{F}^Y)^{\text{an}}$ are isomorphic on the stalks, and Proposition 2.2.12 of [4] therefore implies that the two sheaves are isomorphic.

$\square$

Note that $\mathcal{F}^Y$ still remains coherent, and Propositions 2 implies that $(\mathcal{F}^{\text{an}})^Y \cong (\mathcal{F}^Y)^{\text{an}}$ will also be a coherent sheaf.

3 First Theorem in GAGA

We will continue with the notation that we've been using throughout this paper. That is, let X be an algebraic variety with a coherent sheaf $\mathcal{F}$, and let X^{an} and $\mathcal{F}^{\text{an}}$ be their analytificiations, respectively. Suppose that U is a closed subset of X and consider a section $s \in \mathcal{F}(U)$. Then using the map $\phi : X^{\text{an}} \to X$, we can consider s as a section s' of the inverse image functor $\phi^{-1}\mathcal{F}$ over the open $U^{\text{an}} \subset X^{\text{an}}$. With this, we can construct the the map $\epsilon : s \mapsto s' \otimes 1$ sending s to a a section of $\phi^{-1}\mathcal{F} \otimes \mathcal{H}_{X^{\text{an}}}$ over U^{an}, which induces the mapping

$$\epsilon : \mathcal{F}(U) \longrightarrow \mathcal{F}^{\text{an}}(U^{\text{an}})$$

This map extends to a map on Čech cohomology in a natural way; suppose that $\mathcal{U} = \{U_i\}$ is a finite open cover of X and $\mathcal{U}^{\text{an}} = \{U_i^{\text{an}}\}$ is the corresponding finite open cover of X^{an}. Then for any combination of indices $i_0, \ldots, i_q$, what we said above implies that we have a homomorphism

$$\epsilon : \mathcal{F}(U_{i_0} \cap \ldots, \cap U_{i_q}) \longrightarrow \mathcal{F}^{\text{an}}(U_{i_0}^{\text{an}} \cap \ldots, \cap U_{i_q}^{\text{an}})$$

which induces a homomorphism on the q-cochains

$$\epsilon : C^q(\mathcal{U}, \mathcal{F}) \longrightarrow C^q(\mathcal{U}^{\text{an}}, \mathcal{F}^{\text{an}})$$

for every positive integer q. If we let d_q denote the differential on the cochains $C^q(\mathcal{U},\mathcal{F})$ and d_q^{an} denote the differential on the cochains $C^q(\mathcal{U}^{\text{an}},\mathcal{F}^{\text{an}})$, then we have $d_p^{\text{an}}\circ\epsilon=\epsilon\circ d_p$ from basic properties of the tensor product. That is, our map ϵ commutes with the differential, so it is a cochain map and induces a map on the sheaf cohomology

$$\epsilon : H^q(\mathcal{U},\mathcal{F}) \longrightarrow H^q(\mathcal{U}^{\text{an}},\mathcal{F}^{\text{an}})$$

Finally, taking the direct limit over the open covers, we get a mapping on the Čech cohomology groups

$$\epsilon : H^q(X,\mathcal{F}) \longrightarrow H^q(X^{\text{an}},\mathcal{F}^{\text{an}})$$

With this setup, we have the first major theorem of Serre's *Géométrie Algébrique et Géométrie Analytique*:

Theorem 1: If X is a projective variety with a coherent algebraic sheaf $\mathcal{F}$, then for every integer $q\geq 0$, the homomorphism

$$\epsilon : H^q(X,\mathcal{F}) \longrightarrow H^q(X^{\text{an}},\mathcal{F}^{\text{an}})$$

defined above is bijective.

To show why this is true, we will reduce to the case of a simple algebraic variety, and then prove the result in two cases (as Lemmas below) before generalizing to an arbitrary one. First, note that for any projective variety X, there is a closed embedding $X\to\mathbb{P}^r(\mathbb{C})$ for some integer r. For any coherent sheaf $\mathcal{F}$ on X, we can extend it to a coherent sheaf $\mathcal{F}^{\mathbb{P}^r(\mathbb{C})}$ on $\mathbb{P}^r(\mathbb{C})$ which is zero on $\mathbb{P}^r(\mathbb{C})\smallsetminus X$. From Proposition 8 in Section 26 of [5], we have isomorphisms

$$H^q(X,\mathcal{F})\cong H^q(\mathbb{P}^r(\mathbb{C}),\mathcal{F}^{\mathbb{P}^r(\mathbb{C})}) \quad \text{and} \quad H^q(X^{\text{an}},\mathcal{F}^{\text{an}})\cong H^q\big(\mathbb{P}^r(\mathbb{C})^{\text{an}},(\mathcal{F}^{\text{an}})^{\mathbb{P}^r(\mathbb{C})}\big)$$

From the naturally of the map ϵ, we have the following commutative diagram

$$\begin{tikzcd}
H^q(X,\mathcal{F}) \arrow[r, "\epsilon"] \arrow[d, "\cong"'] & H^q(X^{\text{an}},\mathcal{F}^{\text{an}}) \arrow[d, "\cong"] \\
H^q(\mathbb{P}^r(\mathbb{C}),\mathcal{F}^{\mathbb{P}^r(\mathbb{C})}) \arrow[r, "\epsilon"] & H^q\big(\mathbb{P}^r(\mathbb{C})^{\text{an}},(\mathcal{F}^{\mathbb{P}^r(\mathbb{C})})^{\text{an}}\big)
\end{tikzcd}$$

where we are using Proposition 3 to makes sense of the cohomology group in the bottom right of the diagram. Therefore, it suffices to prove Theorem 1 for the case when $X=\mathbb{P}^r(\mathbb{C})$. For the remainder of the section, we will set $X=\mathbb{P}^r(\mathbb{C})$.

Lemma 1: Theorem 1 holds for the structure sheaf $\mathcal{O}_X$.

Proof: By Lemma 5.3.1 in [4], we know that

$$H^q(X, \mathcal{O}_X) \cong \begin{cases} \mathbb{C} & q = 0 \\ 0 & \text{else} \end{cases}$$

On the other hand, by Dolbeault's theorem we know that $H^{p,q}(X^{\text{an}}) \cong H^q(X^{\text{an}}, \Omega^p)$ where Ω^p is the sheaf of holomorphic p forms on M. Since $\mathcal{O}_X^{\text{an}} = \Omega^0$, we can use an explicit calculation of the Dolbeault cohomology (for example found on the Wikipedia page [1]) to get

$$H^q(X^{\text{an}}, \mathcal{O}_X) \cong H^{0,q}(X^{\text{an}}) \cong \begin{cases} \mathbb{C} & q = 0 \\ 0 & \text{else} \end{cases}$$

which proves our result.

$\square$

Lemma 2: Theorem 1 holds for the coherent sheaf $\mathcal{O}_X(n)$.

Proof: We will proceed by induction on r, the dimension of X. The case when $r = 0$ is trivial, so for our inductive hypothesis, suppose that the result holds true for dimensions less than r. Using the homogenous coordinates $t_0, \dots, t_r$, let's define $E \cong \mathbb{P}^{r-1}(\mathbb{C})$ to the the zero lotus $t_r = 0$. Then we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{\times t_r} \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0$$

where $\mathcal{O}_X \to \mathcal{O}_E$ is the restriction mapping, and $\times t_r$ is multiplication by t_r. Now, we can tensor by $\mathcal{O}_X(n)$ to get the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_E(n) \longrightarrow 0$$

Now taking the long exact sequence induced by Čech cohomology, as well as using the naturally of our map ϵ, we get the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & H^q(X, \mathcal{O}_X(n-1)) \arrow[r] \arrow[d, "\epsilon"] & H^q(X, \mathcal{O}_X(n)) \arrow[r] \arrow[d, "\epsilon"] & H^q(X, \mathcal{O}_E(n)) \arrow[r] \arrow[d, "\epsilon"] & H^{q+1}(X, \mathcal{O}_X(n-1)) \arrow[r] \arrow[d, "\epsilon"] & \dots \\
\dots \arrow[r] & H^q(X^{\text{an}}, \mathcal{O}_X(n-1)^{\text{an}}) \arrow[r] & H^q(X^{\text{an}}, \mathcal{O}_X(n)^{\text{an}}) \arrow[r] & H^q(X^{\text{an}}, \mathcal{O}_E(n)^{\text{an}})) \arrow[r] & H^{q+1}(X^{\text{an}}, \mathcal{O}_X(n-1)^{\text{an}}) \arrow[r] & \dots
\end{tikzcd}$$

We can identify the cohomology groups $H^q(X, \mathcal{O}_E(n)) \cong H^q(E, \mathcal{O}_E(n))$ and $H^q(X^{\text{an}}, \mathcal{O}_E(n)^{\text{an}})) \cong H^q(E^{\text{an}}, \mathcal{O}_E(n)^{\text{an}}))$, so our induction hypothesis implies that

$$\epsilon : H^q(X, \mathcal{O}_E(n)) \longrightarrow H^q(X^{\text{an}}, \mathcal{O}_E(n)^{\text{an}}))$$

is bijective for each n and q.

To conclude our proof, we need to use induction on n. We know the result is true for the base case $n = 0$ by Lemma 1. If our result holds for $\mathcal{O}_X(n-1)$, that is, if

$$\epsilon : H^q(X, \mathcal{O}_X(n-1)) \longrightarrow H^q(X^{\text{an}}, \mathcal{O}_X(n-1)^{\text{an}})$$

is bijective for each q, then the Five Lemma applied to our commutative diagram above implies that

$$\epsilon : H^q(X, \mathcal{O}_X(n)) \longrightarrow H^q(X^{\mathrm{an}}, \mathcal{O}_X(n)^{\mathrm{an}})$$

is bijective.

□

Now we are ready to address Theorem 1 in its full generality.

Proof of Theorem 1: We will proceed by descending induction on q, the q-th Čech cohomology group. As a base case, note that for a coherent algebraic sheaf $\mathcal{F}$ on X, we know that $H^p(X, \mathcal{F}) = 0$ and $H^p(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}) = 0$ for every $p > 2r$ by Grothendieck's vanishing theorem [7].

Now for our inductive case, consider some coherent algebraic sheaf $\mathcal{F}$ on X. By the Corollary to Theorem 1 of Section 55 in [5], there exists a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{O}_X(n)^q \longrightarrow \mathcal{F} \longrightarrow 0$$

Now we can take the long exact sequence induced by Čech cohomology and use the naturally of our map ϵ to get the following commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & H^q(X,\mathcal{R}) \arrow[r] \arrow[d, "\epsilon_1"] & H^q(X,\mathcal{O}_X(n)^q) \arrow[r] \arrow[d, "\epsilon_2"] & H^q(X,\mathcal{F}) \arrow[r] \arrow[d, "\epsilon_3"] & H^{q+1}(X,\mathcal{R}) \arrow[r] \arrow[d, "\epsilon_4"] & H^{q+1}(X,\mathcal{O}_X(n)^q) \arrow[r] \arrow[d, "\epsilon_5"] & \dots \\
\dots \arrow[r] & H^q(X^{\mathrm{an}},\mathcal{R}^{\mathrm{an}}) \arrow[r] & H^q(X^{\mathrm{an}},(\mathcal{O}_X(n)^q)^{\mathrm{an}}) \arrow[r] & H^q(X^{\mathrm{an}},\mathcal{F}^{\mathrm{an}}) \arrow[r] & H^{q+1}(X^{\mathrm{an}},\mathcal{R}^{\mathrm{an}}) \arrow[r] & H^{q+1}(X^{\mathrm{an}},(\mathcal{O}_X(n)^q)^{\mathrm{an}}) \arrow[r] & \dots
\end{tikzcd}$$

From Lemma 2, we know that the maps e_2 and e_5 are bijective, and from our inductive hypothesis we also get that e_4 is bijective. By one of the Four-Lemmas (the first one listed on wikipedia), we get that e_3 must be surjective. Furthermore, since we have shown this for an arbitrary coherent sheaf $\mathcal{F}$, the same must hold for $\mathcal{R}$, which implies that e_1 is also surjective. Therefore, by the 5-Lemma, we get that e_3 must be a bijection.

□

4 Second Theorem of GAGA

Now let us cover the next part of Serre's paper. The following results are the remaining major theorems from GAGA:

Theorem 2: Morphisms between two coherent algebraic sheaves, $\mathcal{F}$ and $\mathcal{G}$ on X, are in bijection with morphisms between the coherent analytic sheaves, $\mathcal{F}^{\mathrm{an}}$ and $\mathcal{G}^{\mathrm{an}}$.

Theorem 3: For every coherent analytic sheaf $\mathcal{M}$ on X^{an}, there is a unique coherent algebraic sheaf $\mathcal{F}$ on X such that $\mathcal{M}$ is isomorphic to $\mathcal{F}^{\text{an}}$.

Equivalently, Theorem 2 states that the functor $\mathcal{F} \mapsto \mathcal{F}^{\text{an}}$, between the category of coherent algebraic sheaves and the category of coherent analytic sheaves, is fully faithful. Likewise, in the language of category theory, Theorem 3 means that the functor is essentially surjective. These two theorems combined means that there is an equivalence of categories between coherent algebraic sheaves and coherent analytic sheaves.

Theorem 3 takes a little more work than we have time for here, but Theorem 2 is simple enough to show. First, given a sheaf of rings $\mathcal{R}$, and two $\mathcal{R}$-modules, $\mathcal{F}$ and $\mathcal{G}$, we define the *sheaf of germs of homomorphisms* to be the sheaf which maps $U \mapsto \text{Hom}_{\mathcal{R}(U)}(\mathcal{F}(U), \mathcal{G}(U))$, and we denote it as $\text{Hom}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$. It can be shown (for instance in Proposition 5 of Section 14 in [5]) that if $\mathcal{F}$ is a coherent sheaf, then we have an isomorphism $\big(\text{Hom}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})\big)_x \cong \text{Hom}_{\mathcal{R}_x}(\mathcal{F}_x, \mathcal{G}_x)$. With these tools, we are ready to prove our next theorem.

Proof of Theorem 2: Let $\mathcal{F}$ and $\mathcal{G}$ be two coherent algebraic sheaves on X, and set $\mathcal{A} = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $\mathcal{B} = \text{Hom}_{\mathcal{H}_{X^{\text{an}}}}(\mathcal{F}^{\text{an}}, \mathcal{G}^{\text{an}})$. Our first goal is to show that there is an isomorphism between $\mathcal{A}^{\text{an}}$ and $\mathcal{B}$, and by Proposition 2.2.12 in [4], it suffices to show there is an isomorphism on the stalks. Recall that from the continuous inclusion $\phi : X^{\text{an}} \to X$, we get that $(\phi^{-1}\mathcal{A})_x = \mathcal{A}_x$ for all $x \in X^{\text{an}}$, and similarly for any coherent sheaf on X.

Given any $x \in X^{\text{an}}$, an element $\psi \in \mathcal{A}_x = \text{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$ is simply a morphism from $\mathcal{F}_x$ to $\mathcal{G}_x$. Therefore, we can construct a map

$$\begin{aligned}(\phi^{-1}\mathcal{A})_x &\longrightarrow \mathcal{B}_x \\ \psi &\longmapsto \psi'\end{aligned}$$

where $\psi' : \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_{X^{\text{an}}} \to \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_{X^{\text{an}}}$ is the map which sends $\psi' : f \otimes h \mapsto \psi(f) \otimes h$. Finally, applying the functor $\otimes_{\mathcal{O}_{X,x}} \mathcal{H}_{X^{\text{an}},x}$ we get a map

$$\iota_x : \mathcal{A}_x^{\text{an}} \longrightarrow \mathcal{B}_x$$

As stated above, note that this is actually a map

$$\iota_x : \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_{X^{\text{an}},x} \longrightarrow \text{Hom}_{\mathcal{H}_{X^{\text{an}},x}}(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_{X^{\text{an}},x}, \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{H}_{X^{\text{an}},x})$$

Since $(\mathcal{O}_{X,x}, \mathcal{H}_{X^{\text{an}},x})$ is a flat couple, Proposition 21 in [6] implies that this is actually an isomorphism.

Now consider the homomorphisms

$$H^0(X, \mathcal{A}) \xrightarrow{\epsilon} H^0(X^{\text{an}}, \mathcal{A}^{\text{an}}) \xrightarrow{\iota} H^0(X^{\text{an}}, \mathcal{B})$$

Note that an element of $H^0(X, \mathcal{A})$ is a global section of $\mathcal{A}$, which is simply a morphism from $\mathcal{F}$ to $\mathcal{G}$. Similarly, $H^0(X^{\text{an}}, \mathcal{B})$ is the group of morphisms from $\mathcal{F}^{\text{an}}$ to $\mathcal{G}^{\text{an}}$. Theorem 1 implies that ϵ is bijective, and our statement above means that ι is bijective, so we have a bijective map from $H^0(X, \mathcal{A})$ to $H^0(X^{\text{an}}, \mathcal{B})$.

$\square$

Acknowledgements

I would like to extend my deepest gratitude to Professor Sophie Morel, whose guidance and insight made it a pleasure to start learning algebraic geometry. I'd also like to thank Gary Hu, who helped make this reading course possible.

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